

Expressiveness of Elementary Cellular Automata

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We investigate expressiveness, a parameter of one-dimensional cellular automata, in the context of simulated biological systems. The development of elementary cellular automata is interpreted in terms of biological systems, and biologically inspired parameters for biodiversity are applied to the configurations of cellular automata.

This article contains a survey of the Elementary Cellular Automata in terms of their expressiveness and an evaluation whether expressiveness is a meaningful term in the context of simulated biology.

1. Introduction

Expressiveness is a parameter for cellular automata that was introduced by Andrew Adamatzky and Leon O. Chua¹ in the context of two-dimensional cellular automata for the simulation of memristive networks². It is intended to measure the “complicatedness” of a configuration of a cellular automaton. This article, together with reference 3, is part of an ongoing project to find measures for the complexity of the behaviour of cellular automata.

We are interested in complexity measures for cellular automata that simulate technological devices or naturally occurring systems. Expressiveness was introduced for simulated memristive networks; now we explore it in the context of simulated biological systems: We view the configuration of the cellular automaton as a biotope

and the local neighborhoods of the cells as corresponding to the individuals of a species. A complex configuration of the cellular automaton – with many different kinds of neighborhoods – then corresponds to a species-rich biotope. We want to apply indices of biodiversity of the biological literature to each of the configurations of the development of a cellular automaton over time and compare them so with biological systems and their behavior.

Why this indirect approach? A more natural method, often used, is to use each of the cell states of a cellular automaton to simulate an individual of a different species, such that the fact that a cell is in state σ means that an individual of species σ is at the location symbolized by the cell. This requires at least n cell states for n species, and the large number of possible transition rules makes it impossible to test them all experimentally.

Instead we restrict our attention to the small number of elementary cellular automata, but we interpret them differently. We assume

- (1) that individuals of different species prefer different environments,
- (2) that each individual is located at one location (or *cell*) of the cellular automaton, and
- (3) that the environment that determines which species lives at a certain location is the neighborhood of that location.

We may then choose a neighborhood size of n cells. With it we can characterize, in a cellular automaton with 2 states per cells, the development of up to 2^n species in a biotope. And on the other hand, the small number of elementary cellular automata allows to investigate the behavior for all of them easily.

2. Notation

Elementary Cellular Automata An *elementary cellular automaton* (ECA) consists of a *state set* $\Sigma = \{0, 1\}$ and a *local transition rule* $\varphi: \Sigma^3 \rightarrow \Sigma$.

The *configurations* of the ECA are the function from \mathbb{Z} to Σ , the set of all configurations is therefore $\Sigma^{\mathbb{Z}}$. A finite sequence of cell states in a configuration, like “0001000”, is a *pattern*.

The *evolution* of a cellular automaton is an infinite sequence $(c_t)_{t \geq 0}$ of configurations, such that each configuration c_t determines its successor configuration c_{t+1} by the *global transition rule*

$$c_{t+1}(x) = \varphi(c_t(x-1), c_t(x), c_t(x+1)) \quad \text{for all } x \in \mathbb{Z}. \quad (1)$$

The configuration c_0 is the *initial configuration* of the evolution.

Code Numbers There are 256 possible transition rules; they are – as usual – referred to by their *code numbers*, popularized by Stephen Wolfram.⁴ One writes the values of a transition rule φ as a sequence

$$\varphi(1, 1, 1)\varphi(1, 1, 0)\varphi(1, 0, 1)\varphi(1, 0, 0)\varphi(0, 1, 1)\varphi(0, 1, 0)\varphi(0, 0, 1)\varphi(0, 0, 0) \quad (2)$$

and interprets the resulting series of zeros and ones as binary number. The terms in (2) are also arranged by their value of the local neighborhood x_1, x_2, x_3 in $\varphi(x_1, x_2, x_3)$ when it is interpreted as a binary number.

One example: The number 127, written in binary, has the form 10000000. This means that the elementary cellular automaton with the code number 127 has a transition rule φ_{127} with $\varphi_{127}(1, 1, 1) = 1$ and $\varphi_{127}(x_1, x_2, x_3) = 0$ for all other values of $x_1, x_2, x_3 \in \Sigma$.

3. Types of Generative Behavior

3.1. *Experimental Setup*

As a test for the behavior of the cellular automata in general, we are interested in the *generative behavior* of cellular automata. This is a scenario in which all cells of the initial configuration except those in a finite region are in state 0. The content of the finite region – the *seed* – is then kept fixed and its evolutions under all transition rules are compared. The investigation of seed patterns is a common research method^{3,5,6}.

In the biological context this is a scenario where we have a single seed in an otherwise barren landscape. We will use the smallest possible seed pattern, a single cell in state 1. This configuration is evolved over 200 time steps for every elementary cellular automaton rule up to equivalence – which will be explained next.

We will use a 3-cell neighborhood to define the species in the biological interpretation. One of these neighborhoods, the pattern 000, represents according to our interpretation an uninhabited location. In the calculation of the biological diversity indices it is therefore left out and not counted as a species.

With this experimental setup we have therefore 7 species and a desert – just enough to expect some nontrivial interactions.

Finding a Sample of ECA Rules When surveying the behavior of a set of one-dimensional cellular automata, like the ECA, one usually does not distinguish a rule from another one in which left and right are exchanged (its *reflection*), or from one in which the states 0 and 1 are exchanged (its *negative*).^a Thus a transition rule can be equivalent to up to three other rules. From them one chooses usually the cellular automaton rule with the lowest number to represent the equivalence class (see e. g. Ref 9).

One effect of this selection is that there are no rules in the sample with $\varphi(0, 0, 0) = \varphi(1, 1, 1) = 1$. For such a rule, its negative would be a rule φ' with $\varphi'(1, 1, 1) = \varphi'(0, 0, 0) = 0$, which has a lower code number than φ . Therefore φ can not be part of the sample.

In our context, where we have an initial configuration $\dots 0001000 \dots$, the cell states 0 and 1 become however distinguishable, and at most two rules can be really

^aWe use here the terminology of Andrew Wuensche as described in Ref 7.

equivalent. On the other hand, the interpretation of $\dots 0000\dots$ as desert would require that we only use rules in which $\varphi(0,0,0) = 0$, so that the desert stays unchanged. We will however extend the set of rules to include rules that have $\varphi(0,0,0) = 1$ and $\varphi(1,1,1) = 0$. In them, an empty background configuration $\dots 00000\dots$ will evolve to $\dots 11111\dots$ and back, and these two configurations oscillate forever. We call these rules here the *flickering* rules. Among the rules shown in Figure 1, Rules 45, 57 and 73 are flickering.

In a literal interpretation of the scenario above, these rules would have no quiescent background of zeroes on which a finite pattern could evolve. We can however look at the evolution of such a cellular automaton only at every second step, when all background cells are in state 0. We can then again interpret the configuration $\dots 00000\dots$ as an inactive background, and the flickering rules are no longer a special case.

This is the reason why the evolution of the cellular automata in the experiments runs over an even number of time steps.

3.2. Qualitative Behavior of Seeds

To interpret the results, we use a simple phenomenological classification of the cellular evolutions arising from a one-cell seed. We have found five types and one subtype.

- (1) *Evolution to zero (0)*. After one time step, the seed cell has vanished and the background remains.
- (2) *Finite Growth (F)*. The seed cell develops into a periodic pattern of finite size.
- (3) *Periodic patterns (P)*. In them, the zone of influence of the “seed cell” in state 1 consists of the repetition of a simple pattern. In Rule 50, this is the pattern “01010101...”.
 - (a) A subtype of the periodic patterns are the *bipartite periodic patterns (P2)*. In them, the left side and the right side of the zone of influence have different patterns. There are only three examples of this behavior, Rules 57, 62 and 109.
- (4) *Sierpiński patterns (S)*. These are the patterns in which a fractal-like structure arises.
- (5) *Complex behavior (C)*. In these patterns no simple structure is visible. More complex structures, less easy to describe, are possible.

The letters in brackets are the abbreviated names of the types, for use in tables and diagrams. Examples for the types can be seen in Figure 1.

One may notice the similarity of this scheme to the wellknown classification of cellular automata by Wolfram.⁸ It is inspired by it but nevertheless has to be kept separate. This is because Wolfram’s classification relies on the behaviour of the cellular automaton for random initial configurations, while the current scheme uses only one specific configuration. As a result of this difference we get types of

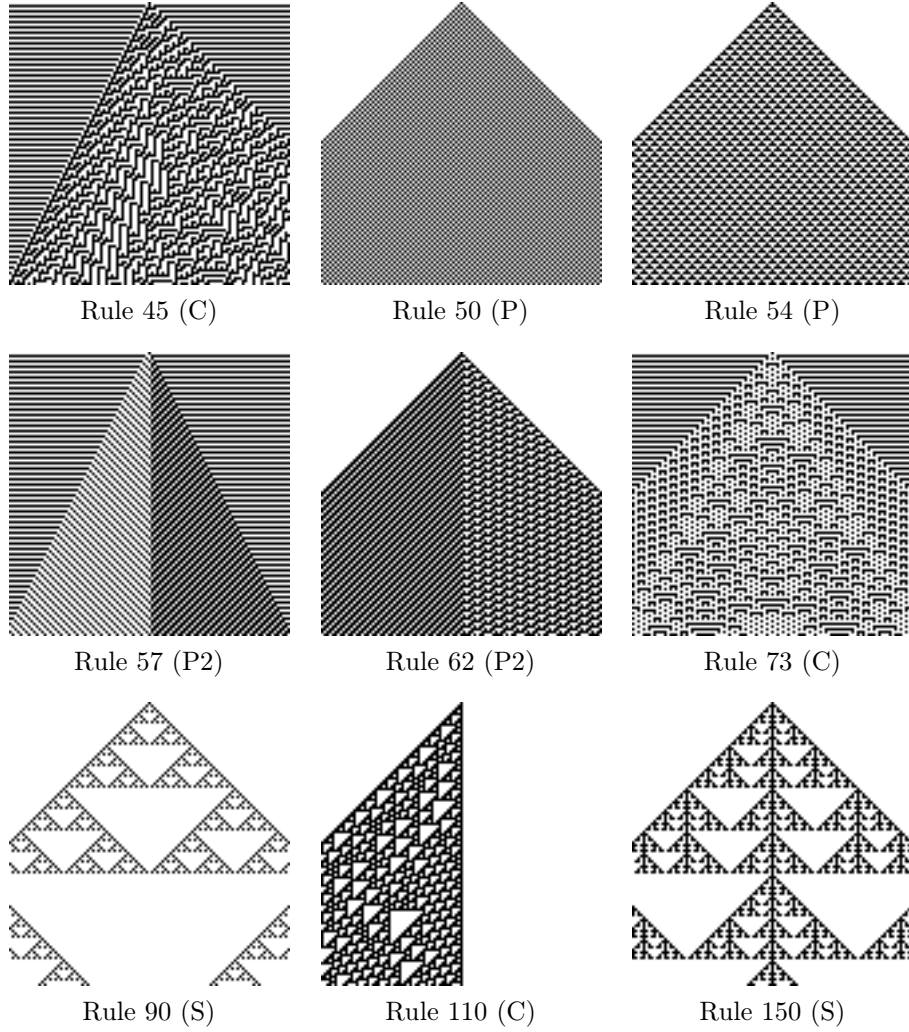


Figure 1. Phenomenological types of evolution for the one-cell seed configuration: Examples for periodic (P and P2), complex (C) and Sierpiński (S) evolution.

behaviour that have no clear equivalent to Wolfram's classes, namely P, P2 and S. In them the behaviour of the cellular automaton is much more regular than with random initial configurations. The other types, 0, F and C, correspond in their definition to Wolfram's classes 1, 2, and 4 respectively, but they are different as sets. One example is Rule 54, which belongs to Wolfram's class 4 but here has type P, not C. Therefore it is recommendable to use here a classification scheme that is visibly different from Wolfram's.

3.3. Diversity Parameters

We now define and explain expressiveness and related parameters for the context of cellular automata. Expressiveness is defined with the help of entropy, so we must define entropy first.

We write $\#_p(c)$ for the number of occurrences of a pattern $w \in \Sigma^*$ in a configuration $c \in \Sigma^{\mathbb{Z}}$. Thus $\#_1(c)$ is the number of cells in state 1 in the configuration c .

The number $\#_w(c)$ may be infinite. If $\#_1(c)$ is finite, then $\#_w(c)$ is also finite for all $w \in \Sigma^*$ that do not consist entirely of zeros. Therefore, in the following definitions, we will always require that $\#_1(c) < \infty$.

In the evolution of a seed configuration, this condition is always fulfilled for rules with $\varphi(0,0,0) = 0$, and for flickering rules it is true at every even-numbered time step.

Entropy The *entropy* (or “Shannon Entropy” in the biological literature; see Ref 10.) is an often used parameter to measure diversity in a biological system. In biology, it is given by the formula

$$H(c) = - \sum_{w \in W} \nu_w(c) \ln \nu_w(c), \quad (3)$$

where c is the biotope, W the set of species in the biotope and ν_w the fraction of individuals of species w among the total population of the biotope.

In the context of cellular automata, the biotope becomes the configuration c and the set of species is replaced with $W = \Sigma^3 \setminus \{(0,0,0)\}$, the set of inhabited 3-cell neighborhoods – $(0,0,0)$ is by definition uninhabited. The *relative frequency* of $w \in W$ in c is then

$$\nu_w(c) = \frac{\#_w(c)}{\sum_{w \in W} \#_w(c)}. \quad (4)$$

In this expression, the numerator is finite for all $w \in W$ if and only if $\#_1(c) < \infty$. The denominator is not 0 if and only if $\#_1(c) > 0$. Therefore $\nu_w(c)$ is defined for all $w \in W$ if $0 < \#_1(c) < \infty$, and the same is true for $H(c)$.

Expressiveness The space-filling ratio and the expressiveness of a configuration are only defined for configurations that arise from a seed pattern; we must know the time that has passed since the evolution of the seed started. Let therefore c be the configuration that arises a time step t of the evolution of a seed.

In an elementary cellular automaton, the state of a cell depends only on the states of the same cell and its direct neighbours at the previous time step. Therefore a growing structure on a quiescent background can grow at every time step by maximally one cell to the left and one cell to the right. This means that a single cell in state 1 can cause at time step t at most $2t + 1$ to be cells in state 1. The *space-filling ratio* of c at time t is therefore the number

$$\rho_t(c) = \frac{\#_1(c)}{2t + 1}. \quad (5)$$

The *expressiveness* of the configuration c at time t is the ratio

$$e_t(c) = \frac{H(c)}{\rho_t(c)}. \quad (6)$$

Like $H(c)$, the expressiveness $e_t(c)$ exists if and only if $0 < \#_1(c) < \infty$.

4. Testing the Model

4.1. The Species-Area Relation

Theory In many cases there exists a simple relation between the area of a region and the number of species it contains. This *species-area relation* is a power law the form

$$S = \gamma A^z, \quad (7)$$

where S is the number of species in a region and A is its size. “The slope of the relationship, z , is commonly found to be about 0.25 to 0.30 (although values span the range 0 to 0.5)”¹¹. We can use this relationship as a measure for the validity of our biological interpretation for cellular automata.

If the identification of cellular neighborhoods with species is correct for a cellular automaton, then a relationship of the form (7) should exist for it. In this context, the *number of species* becomes the number of different neighborhoods found in a configuration. For the *area* of the biological system we take, in first approximation, the length of the interval from the leftmost to the rightmost cell in state 1.^b More precisely, we define the *active zone* of a configuration c as the shortest interval $[x_0, x_1]$ such that for all positions $x \in \mathbb{Z}$ with $c(x) = 1$ we have $x_0 \leq x \leq x_1$. The *width* of c is then the number $w = x_1 - x_0 + 1$.

We take the possibility of boundary effects into account and make not w but $w + \delta$ our equivalent to the area in (7). The constant δ is the size of the boundary zone. If it is greater than 0, then the actually inhabited area is larger than the active zone. With this change, the species-area relation for one-dimensional cellular automata becomes

$$S = \gamma(w + \delta)^z, \quad (8)$$

where S is the number of neighborhoods and c , γ and δ are constants that depend on the cellular automaton.

A case where δ comes into play are the completely chaotic configurations – those in which the states of all neighborhoods that overlap with the active zone are

^bThere are other intuitively meaningful interpretations of “area” in this context, e.g. the number of cells in state 1 in a configuration. We have tried this and got similar results, but the data were more scattered than in Figure 2. We therefore prefer the current definition, hoping to get a simpler theory with it.

different. In this case, if the length of the neighborhood is ℓ , we have $\gamma = z = 1$ and $\delta = \ell + 1$, and the species-area relation has the form

$$S = w + \ell - 1. \quad (9)$$

To see this, we assume that the active zone consists of the cells at $x_0, x_0 + 1, \dots, x_1$, as before. A neighborhood of length ℓ with leftmost point ξ reaches from ξ to $\xi + \ell - 1$. The leftmost neighborhood that overlaps with the active zone has therefore the leftmost point $\xi_0 = x_0 - \ell + 1$, the rightmost overlapping neighborhood has the leftmost point $\xi_1 = x_1$, and so there are $\xi_1 - \xi_0 + 1 = x_1 - x_0 + \ell = w - 1 + \ell$ neighborhoods. If their states are all different, their number is given by (9).

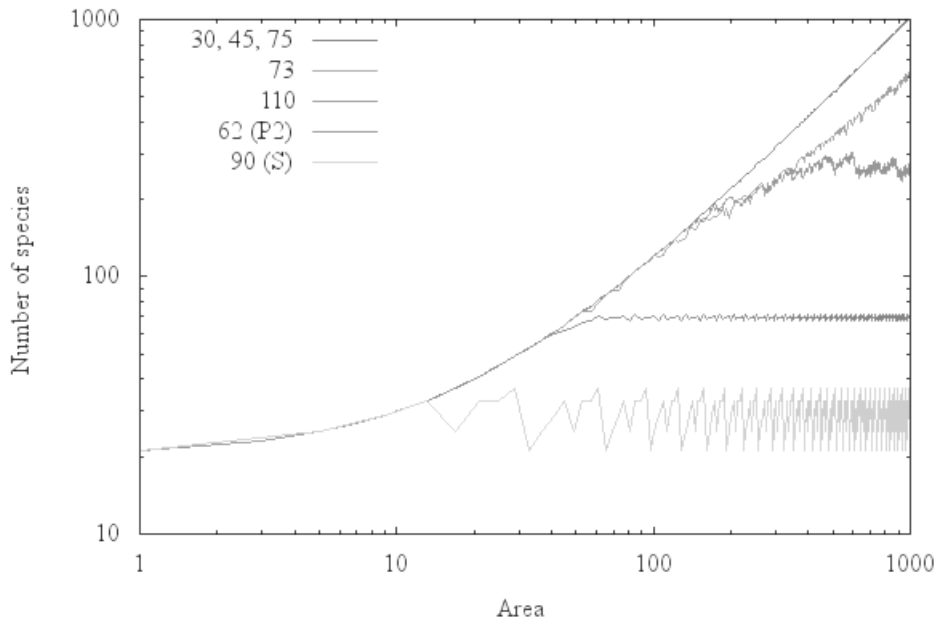


Figure 2. Species-Area relations for complex and other rules. Only odd time steps are considered. Rules 30, 45 and 75 create graphs that are indistinguishable.

Results Figure 2 shows the relation between the number of neighborhoods and area for several rules. To verify the relationship (7) experimentally one needs a large number of possible species, so we have used here a neighborhood size of 20, which gives us $2^{20} = 1048576$ possible species. This number is not exhausted by the cellular automata.

If the number of neighborhoods in Figure 2 stays bounded, this must therefore be caused by the rule and not by the sample. There are several rules for which this is must be the case – it is obvious for the types 0 (evolution to zero), F (finite growth),

P and P2 (periodic patterns). In the Sierpiński patterns there is a periodic raise and fall in the number of neighborhoods, but overall their number stays bounded.

This leaves the rules of type C as the only ones where we can expect behavior that simulates biological systems. We see that for rules 30, 45, 73 and 75 the relation between the number of neighbourhoods and the area is asymptotically a power law and appears as a straight line in the diagram. However, in all all rules except 73 the exponent in this power law is near 1, which reduces it to the trivial case of a linear relation. For Rule 110 the number of species finally stays bounded. The cause for this is certainly the presence of the “ether”, a periodic pattern that arises from almost all initial configurations.¹² The ether regions finally dominate the configurations, making them look almost like a periodic pattern.

For the remaining rules, species-area relations can be determined by a least-square approximation of (8). The results are shown in Table 1. We see that the

Table 1. Coefficients for the species-area relation.

Rule code	γ	z	δ
30	0.983	1.002	20.998
45	1.000	1.000	20.013
75	1.001	1.000	19.911
73	1.088	0.905	89.627

exponents are higher than in the biological case: all of the rules have $z \approx 1$. In the first three rules the species-area relation has approximately the form $S = w - 20$, which is almost exactly the relation (9) for chaotic rules. (Rule 30 is famous for being chaotic, see e.g. Rowland¹³.) This leaves us with Rule 73 as the only one with a nontrivial species-area relation, at least when starting with an isolated 1 as a seed.

4.2. The Expressiveness of Transition Rules

We want to use expressiveness as a means to distinguish transition rules by the complexity and richness of the behavior they generate. With the current definition, expressiveness exists however only for configurations, for single moments in time.

If it stabilizes over time, expressiveness becomes a meaningful property of transition rules. We must therefore test whether the value of $e_t(c_t)$ during an evolution $(c_t)_{t \geq 0}$ converges as t goes to infinity.

From now on we will have two concepts of expressiveness that must be clearly distinguished. The first is $e_t(c_t)$, the *expressiveness of a configuration*. The second concept is the *expressiveness of a rule*, or the limiting value of $e_t(c_t)$ as t goes to infinity (if it exists).

As we see in Figure 3, it depends on the type of a rule whether it is meaningful to speak of its expressiveness. It is clearly, in some approximation, a well-defined concept for rules with complex behavior. For all the rules shown in the diagram the

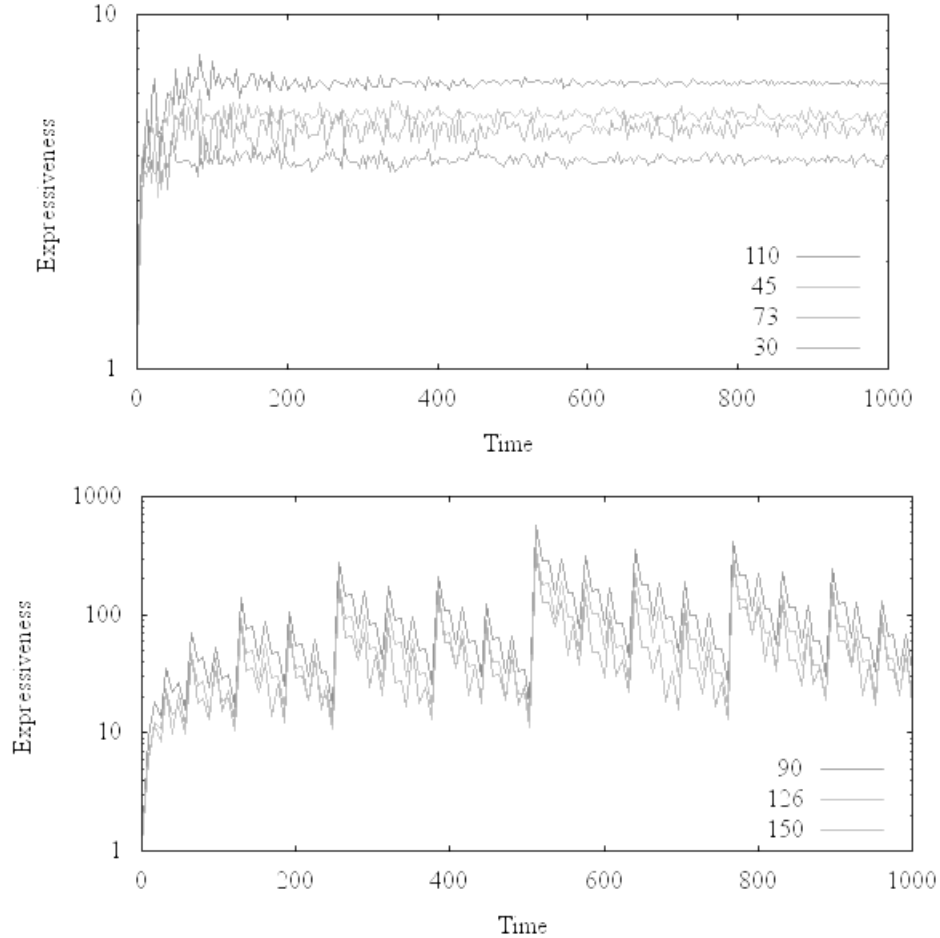
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Figure 3. Change of configuration expressiveness over time. The upper diagram shows some complex rules, the lower diagram rules with Sierpiński-like behavior. To avoid a too cluttered display, in the upper diagram only points with a time coordinate divisible by 4 are shown and in the lower diagram only the points with coordinates divisible by 8. The vertical arrangement of the lines in the plots is the same as that of the numbers in the legend.

expressivenesses of its configurations become very fast restricted to a small interval, and, with the exception of Rule 73, the intervals of different rules do not even intersect. (If we were to need an single number as estimate for the expressiveness of these rules, an average over a few generations, maybe just 10 or 50, might be enough.) On the other hand, the expressiveness of the configurations under a Sierpiński-like rule always varies and never stabilizes at a value. Therefore it is not meaningful to speak of the expressiveness of, say, Rule 90.^c Nevertheless the range of values seems

^cAs the three rules shown in Figure 3 vary in the same way, the relative expressiveness of such a

to be bounded, even if it extends over several orders of magnitudes.

We can also see from Figure 3 that the expressivenesses of the configurations under complex and Sierpiński rules occupy different ranges of numbers. So, even if “rule expressiveness” is not defined for every ECA rule, the expressiveness of a single configuration at a specific time may still be enough to distinguish between rules of different types. This is subject of the following section.

5. Results

5.1. Classification by Expressiveness

Here we investigate whether the expressiveness of a configuration at a specific time step, namely the 200th, does reflect their phenomenological type. The values in Table 2 show that this is indeed the case, with some exceptions.

While expressiveness is not defined for rules of type 0, the main result of Table 2 is that the other phenomenological types, sorted by expressiveness, appear in the order

$$P \prec C \prec S \prec F. \quad (10)$$

with P2 sorted with P and only a few exceptions to a strict ordering.

- (1) $P, P2$: $0 < e_t(c_t) \leq 3.268$ for most rules that generate periodic structures.
- (2) C : $3.794 \leq e_t(c_t) < 8$ for all rules with complex behavior.

If we set the boundaries between the regions for P and C in the way described here, the C region contains three rules with periodic behavior. One is Rule 54. Since it has in general a quite complex behavior,^{15,16} this rule should “rightfully” belong to the complex rules. Yet, while the complex behavior of Rule 54 is not visible from the behavior of the one-cell initial configuration, the expressiveness of the generated pattern is still abnormally high.

The other exceptions are Rule 57 and its color-reversed version, Rule 99. Both are bipartite and periodic. The bipartite structure of their evolution may contribute to its high expressiveness.

- (3) S : $29 < e_t(c_t) < 55$. All Sierpiński rules have high expressiveness. One sees from Table 2 that this is caused by the very low space-filling ratio of these patterns at time step 200. (There are times when the configuration of a Sierpiński pattern consists mainly of cells in state 1, but these are rare; for Rule 90, e. g., they occur when t is a power of 2.)
- (4) F : $107 < e_t(c_t) < 441$ for rules that generate a structure of bounded size. If the growth of the pattern is finite, entropy stays constant (or changes periodically with time), while the space-filling ratio approaches 0; therefore the expressiveness grows without bounds: it is already quite large at time step 200.

rule in relation to, say, Rule 90, might be a meaningful complexity measure.

Table 2. ECA rules sorted by their expressiveness. If a rule number is followed by another number in brackets, then that is the code number of its negative. The “Type” column describes the qualitative behavior of the initial configuration with a single cell in state 1 as in Section 3.2. The “5-Type” column contains the classification according to Oliveira et al.¹⁴ “0” is Null, “fp” is fixed-point behavior, “c2” is two-cycle, “p” is periodic, “co” is complex and “ch” is chaotic behavior. The other columns contain the parameters defined in Section 3.3.

Rule code	Type	5-Type	# ₁	ρ_{200}	H	e_{200}
0, 8, 32, 40, 128, 136, 160, 168	0	0	0	0.000		
7, 19, 23, 31 (7), 55 (19), 63 (3), 95 (5), 127 (1)	0	c2	0	0.000		
72, 104, 200, 232	0	fp	0	0.000		
50, 178	P	c2	201	0.501	0.724	1.445
122	P	ch	201	0.501	0.724	1.445
58, 77	P	fp	201	0.501	0.724	1.445
94	P	p	202	0.504	0.780	1.548
62	P2	p	250	0.623	1.685	2.703
28, 156	P	c2	101	0.252	0.748	2.972
13, 79 (13)	P	fp	101	0.252	0.748	2.972
109 (73)	P2	ch	237	0.591	1.848	3.127
78	P	fp	102	0.254	0.831	3.268
30	C	ch	204	0.509	1.930	3.794
54	P	co	101	0.252	1.099	4.362
73	C	ch	155	0.387	1.854	4.796
75 (45)	C	ch	146	0.364	1.932	5.307
45	C	ch	146	0.364	1.937	5.321
110	C	co	121	0.302	1.841	6.101
57, 99 (57)	P2	fp	101	0.252	1.792	7.113
105, 150	S	ch	15	0.037	1.099	29.370
126	S	ch	16	0.040	1.386	34.744
18, 22, 60, 90, 146	S	ch	8	0.020	1.099	55.068
26, 154	S	p	8	0.020	1.099	55.068
107 (41)	F	p	6	0.015	1.609	107.564
91 (37)	F	c2	5	0.012	1.475	118.301
111 (9)	F	c2	6	0.015	1.831	122.373
103 (25)	F	c2	5	0.012	1.906	152.874
9, 25	F	c2	4	0.010	1.946	195.077
37, 123 (33)	F	c2	3	0.007	1.609	215.128
11, 14, 43, 47 (11), 59 (35), 142	F	c2	2	0.005	1.386	277.952
46	F	fp	2	0.005	1.386	277.952
1, 3, 5, 6, 15, 27, 29, 33, 35, 38, 39 (27), 51, 71 (29), 74, 108, 134	F	c2	1	0.002	1.099	440.544
106	F	ch	1	0.002	1.099	440.544
2, 4, 10, 12, 24, 34, 36, 42, 44, 56, 76, 130, 132, 138, 140, 152, 162, 164, 170, 172, 184, 204	F	fp	1	0.002	1.099	440.544
41	F	p	1	0.002	1.099	440.544

We also note that the gaps in expressiveness between C and S and from S to F are quite large while there is no visible gap in the range of expressivenesses for the P, P2, and C rules. From the viewpoint of expressiveness, the difference between complex rules and periodic ones is thus only one of degree.

In contrast to this, the five-parameter classification of Oliveira et al.¹⁴ is not

mirrored as well in the expressiveness results. One can nevertheless see that the chaotic rules occur more prominently among the rules with higher expressiveness.

5.2. Connection with the Simpson Index

The *Simpson diversity index* $1 - D$ is the probability that two randomly chosen individuals belong to two different species. The number D , as originally proposed by Simpson¹⁷, measures the homogeneity of a population and is defined as

$$D = \sum_{i=1}^k \frac{n_i(n_i - 1)}{N(N - 1)}. \quad (11)$$

In this formula there are k species, n_i is the number of individuals of species i and $N = \sum_{i=1}^k n_i$ is the number of all individuals.

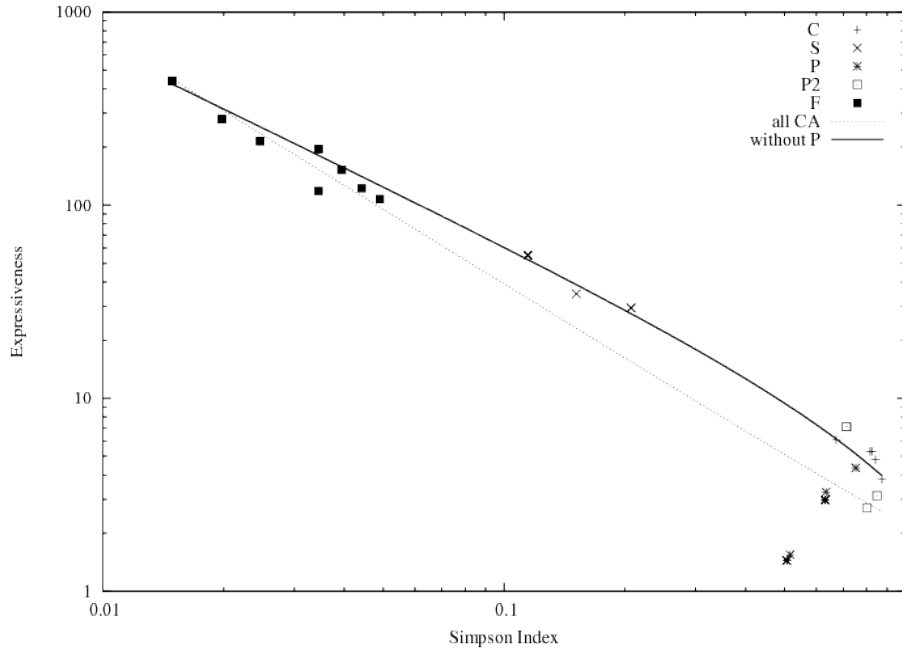


Figure 4. Expressiveness versus Simpson diversity index. The points are marked according to their qualitative behavior, as in Table 2. There are 20 rules that have $e = 0$ or $1 - D = 0$ and therefore do not appear in the diagram.

There is a relation between the Simpson index and expressiveness for elementary cellular automata. Figure 4 is a plot of expressiveness and Simpson Index for all those rules for which both of them are larger than 0. In logarithmic scale it shows that most of them are arranged on a straight line, indicating a power law. To find it, we approximate it by the expression

$$e = A(1 - D)^z + c \quad (12)$$

where A , z and c are constants that must be determined. We do a least square approximation in the context of the double-logarithmic diagram, i. e. the expression

$$\sum_{i=1}^k (\log e_i - \log (A(1 - D_i)^z - c))^2 \quad (13)$$

is minimized. In it, e_i and D_i are the expressiveness and the Simpson index of the i -th ECA rule. The results are $A = 2.0171$, $z = -1.2855$ and $c = 0.1978$.

One can see however from Figure 4 that the periodic rules (P) stand out. If we remove them from the sample, we get $A = 6.3753$, $z = -0.9995$ and $c = -3.3228$. It is remarkable here that z is approximately -1 . So we have for these rules $e \approx A/(1 - D) + c$, or

$$1 - D \approx \frac{A}{e - c}. \quad (14)$$

6. Conclusions

We have looked at different properties of expressiveness. One question was whether expressiveness of a single configuration, when measured at a certain time, can serve as a means to distinguish between cellular automata with different behavior. We have found that this is the case. Another question was whether we can define a numerical value for the expressiveness of a rule. Here we found that this is possible for most of the rules, but not for all of them: For rules that develop Sierpiński patterns, the expressiveness of their configurations varies greatly over time. Nevertheless, its values belong for rules of different phenomenological types to different intervals, so that it is still possible to distinguish the types of the rules by the expressiveness of a sample configuration at a certain time.

So we can characterize the behavior of a cellular automaton by the expressiveness of its configurations. If this automaton simulates a biological system, has this expressiveness then a meaningful interpretation? To answer this question, we investigated whether there is a species-area relation for cellular automata, as there is for many biological systems. Cellular automata with such a relation resemble biological systems most closely. We found that there is only one transition rule that has a non-trivial species-area relation, namely Rule 73. Even in its case, the exponent in the species-area relation is outside the usual range for biological systems. We still have to find a cellular automaton which is realistic also in this respect.

And finally, if expressiveness is a measure for biological diversity in simulated ecosystems, is it related to the diversity indices used in biology? Here we have found, for elementary cellular automata, a surprising empirical connection between the expressiveness of the configuration at the 200th time step and the Simpson index. This suggests that configuration expressiveness is a parameter of a similar kind as the Simpson index and therefore, possibly, as the other diversity measures used in biology.

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